Coordinates and Change of basis

Key Theorem: If $\mathcal{B} = \{b_1, \dots, b_n\}$ is a basis of $\mathbb{R}^n$, then for any $v \in \mathbb{R}^n$, there is a unique way to write v as a linear combination of $b_1, \dots, b_n$.

Proof: Suppose $v = x_1 b_1 + \cdots + x_n b_n$ and $v = y_1 b_1 + \cdots + y_n b_n$.

Then $x_1 b_1 + \cdots + x_n b_n = y_1 b_1 + \cdots + y_n b_n$

$\Rightarrow (x_1 - y_1) b_1 + \cdots + (x_n - y_n) b_n = 0$

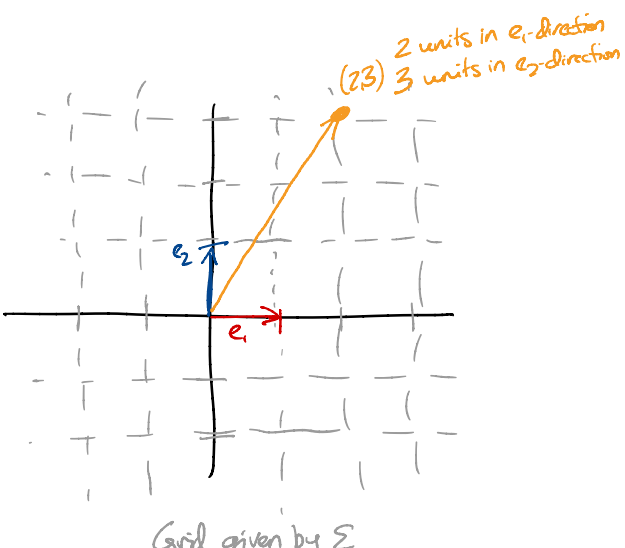
$\Rightarrow x_i - y_i = 0 \quad \forall i$ since $b_1, \dots, b_n$ are L.I.

$\Rightarrow x_i = y_i \quad \forall i$ ∎

If $\mathcal{B} = \{b_1, \dots, b_n\}$ is a basis for $\mathbb{R}^n$ and $v = c_1 b_1 + \cdots + c_n b_n$, then we call $(c_1, \dots, c_n)$ the $\mathcal{B}$-coordinates of v.

Ex: Let $v = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \in \mathbb{R}^n$

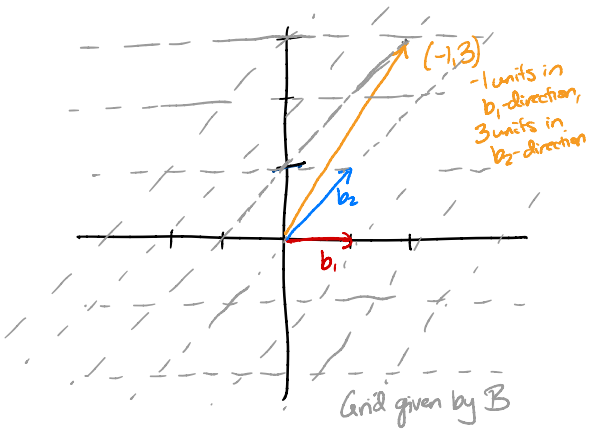
In the standard basis $\mathcal{E} = \{e_1, e_2\}$, we have $v = 2e_1 + 3e_2$. Hence v has $\mathcal{E}$-coordinates $(2, 3)$



Now consider a different basis $\mathcal{B} = \left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}\right\}$

Notice $v = \begin{bmatrix}2\\3\end{bmatrix} = -1\begin{bmatrix}1\\0\end{bmatrix} + 3\begin{bmatrix}1\\1\end{bmatrix}$

Hence v has $\mathcal{B}$-coordinates $(-1, 3)$



Def: If $v \in \mathbb{R}^n$ has $\mathcal{B}$-coordinates $(c_1, \dots, c_n)$, then the **$\mathcal{B}$-coordinate vector** is $[v]_{\mathcal{B}} = \begin{bmatrix}c_1\\ \vdots \\ c_n\end{bmatrix}$.

Ex: In the above example, $[v]_{\mathcal{E}} = \begin{bmatrix}2\\3\end{bmatrix}$ and $[v]_{\mathcal{B}} = \begin{bmatrix}-1\\3\end{bmatrix}$

How do we go from $[v]_{\mathcal{E}}$ to $[v]_{\mathcal{B}}$?

Let $M_{\mathcal{B}} = [b_1 \cdots b_n]$ and $[v]_{\mathcal{B}} = \begin{bmatrix}x_1\\ \vdots \\ x_n\end{bmatrix}$. (ie. $v = x_1 b_1 + \cdots + x_n b_n$)

Then $M_{\mathcal{B}}[v]_{\mathcal{B}} = x_1 b_1 + \cdots + x_n b_n = v = [v]_{\mathcal{E}}$

$M_{\mathcal{B}}$ is called a **change of basis matrix** from $\mathcal{B}$ to $\mathcal{E}$

It takes vectors in the $\mathcal{B}$-basis and rewrites them as vectors in the $\mathcal{E}$-basis

Since $\mathcal{B}$ is a basis, $M_{\mathcal{B}}$ is invertible and $M_{\mathcal{B}}^{-1}[v]_{\mathcal{E}} = [v]_{\mathcal{B}}$

Hence $M_{\mathcal{B}}^{-1}$ changes the $\mathcal{E}$-basis to the $\mathcal{B}$-basis

Ex: $B = \left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}\right\}$ and $v = \begin{bmatrix}2\\3\end{bmatrix} = [v]_\Sigma$ be as above

Then $M_B = \begin{bmatrix}1&1\\0&1\end{bmatrix}$ and $M_B^{-1} = \begin{bmatrix}1&-1\\0&1\end{bmatrix}$

We saw $[v]_B = \begin{bmatrix}-1\\3\end{bmatrix}$. Notice $M_B[v]_B = \begin{bmatrix}1&1\\0&1\end{bmatrix}\begin{bmatrix}-1\\3\end{bmatrix} = \begin{bmatrix}2\\3\end{bmatrix} = [v]_\Sigma$

Now we can use M_B^{-1} to rewrite any vector $v = \begin{bmatrix}x\\y\end{bmatrix}$ in terms of B:

$$[v]_B = M_B^{-1}[v]_\Sigma = \begin{bmatrix}1&-1\\0&1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}x-y\\y\end{bmatrix}$$

If $T: \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then we can associate a matrix A to T by the rule $T(v) = Av \quad \forall v \in \mathbb{R}^n$.

Up until now, we've been implicitly using the standard basis to describe v (and by extension A)

Ex: Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ given by $T\left(\begin{bmatrix}x\\y\end{bmatrix}\right) = \begin{bmatrix}2x+y\\x-y\end{bmatrix}$ (written in Σ-basis)

we have $\begin{bmatrix}2x+y\\x-y\end{bmatrix} = x\begin{bmatrix}2\\1\end{bmatrix} + y\begin{bmatrix}1\\-1\end{bmatrix} = \underbrace{\begin{bmatrix}2&1\\1&-1\end{bmatrix}}_{A}\begin{bmatrix}x\\y\end{bmatrix}$

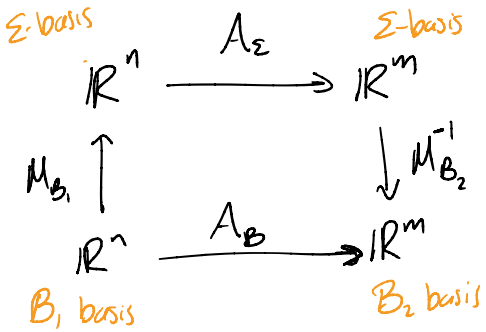
(written in Σ-basis!)

Since each vector is expressed in terms of the standard basis vector, so is A! If we change the basis of $\mathbb{R}^n$, we change what A looks like.

Let A_Σ denote the matrix written in the Σ-basis.

If we change the basis of $\mathbb{R}^n$ to B_1, and the basis of $\mathbb{R}^m$ to B_2 and let A_B be the matrix written in terms of these new bases, then:

$$A_B = M_{B_2}^{-1} A_\Sigma M_{B_1}$$



Ex: Above, $T\left(\begin{bmatrix} x \\ y \end{bmatrix}_\Sigma\right) = \begin{bmatrix} 2x+y \\ x-y \end{bmatrix}_\Sigma$ and $A_\Sigma = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}$

Let's rewrite T in terms of $B = \left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$.

Here, $M_B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $M_B^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

Hence $A_B = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 1 & 0 \end{bmatrix}$

and so $T\left(\begin{bmatrix} x \\ y \end{bmatrix}_B\right) = A_B\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+3y \\ x \end{bmatrix}_B$

Why Change Basis? It can make it easier to understand/work with T if the associated matrix is simpler (e.g. diagonal)